FORMAL MARKOFF MAPS ARE POSITIVE

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ABSTRACT. This note defines a family of Laurent polynomials indexed in $\mathbb{P}^1\mathbb{Q}$ which generalize the Markoff numbers and relate to the character variety of the one-cusped torus. We describe which monomials appear in each polynomial and prove all the coefficients are positive integers. We also conjecture a generalization of that positivity result.

1. INTRODUCTION

In [Bo], Bowditch defined Markoff maps as an appealing way of analyzing the length spectrum of the set $\mathcal{C}$ of simple closed geodesics on a hyperbolic one-cusped torus $S \simeq (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$. He noted that $\mathcal{C}$ stands in natural bijection with $\mathbb{P}^1\mathbb{Q} = \mathbb{Q} \cup \{\infty\}$ via the *slope* function

$$\sigma : \mathcal{C} \xrightarrow{\ \sim\ } \mathbb{P}^1\mathbb{Q} \ ,$$

and associated (bijectively) to each $c \in \mathcal{C}$ a complementary region R_c of an infinite trivalent tree $\mathcal{T}$ properly embedded in the plane. This tree $\mathcal{T}$ is dual to the Farey triangulation of the hyperbolic plane $\mathbb{H}^2$ (see Section 2 for definitions): namely, R_c is the complementary region of $\mathcal{T}$ whose closure in the disc $\mathbb{H}^2 \cup \mathbb{P}^1\mathbb{R}$ contains the ideal point $\sigma(c)$. If $\mathcal{R}$ is the collection of all the regions R_c, the Markoff map

$$\Phi : \mathcal{R} \longrightarrow \mathbb{R}$$

associates to R_c the trace of an element of $SL_2(\mathbb{R})$ representing c (here we choose a lift of the holonomy representation $\pi_1(S) \to PSL_2(\mathbb{R})$). The definition of Φ extends to Kleinian representations $\rho : \pi_1(S) \to SL_2(\mathbb{C})$, and Bowditch studied in particular the relationship between Φ's being proper and ρ's being quasifuchsian. Markoff maps also provide new proofs and generalizations of McShane's identity [Bo, AMS], and their intriguing analytic properties have not yet been fully explored.

Of course, a Markoff map Φ is a very redundant object. It is in fact enough to know $\Phi(R_c)$ for three adjacent regions R_c to reconstruct Φ completely. For instance, denote by R_s the region $R_{\sigma^{-1}(s)}$ for $s \in \mathbb{P}^1\mathbb{Q}$, and consider

$$(1) \qquad \Phi(R_0) = X \ ; \ \Phi(R_\infty) = Y \ ; \ \Phi(R_{-1}) = Z.$$

Then, every $\Phi(R_s)$ can be given by an explicit formula $f_s(X, Y, Z)$. There is in fact a non-trivial algebraic relationship between X, Y, Z, so many very different formulas for f_s exist. In [Gu], we were led to look for expressions of f_s as a Laurent polynomial of degree 1 in X, Y, Z:

$$(2) \qquad f_s = \sum_{\alpha,\beta\in\mathbb{Z}} F_s(\alpha,\beta)\frac{X^{1+\alpha}\,Y^{1+\beta}}{Z^{1+\alpha+\beta}} \ \in \mathbb{Z}\left[X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}\right]$$

In Section 2, we show that such an expression exists, and that furthermore the integer $F_s(\alpha, \beta)$ equals 0 unless (α, β) satisfies a natural parity condition. Our main theorem is

Theorem 1. *The Laurent polynomial f_s has only positive coefficients. Moreover, all monomials in the Newton polygon of f_s which satisfy the parity condition have nonzero coefficients.*

(Recall that the Newton polygon of a Laurent polynomial $P = \sum a_{\nu_1 \ldots \nu_n} X_1^{\nu_1} \ldots X_n^{\nu_n}$ in n variables is the convex hull in $\mathbb{R}^n$ of the points $(\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$ for which $a_{\nu_1 \ldots \nu_n} \neq 0$.) In fact, we describe the Newton polygon of f_s completely (see (4) below). Some examples are shown in Figure 3 page 10. The numbers $f_s(1, 1, 1)$ are the usual Markoff numbers from Diophantine approximation theory [Ca].

The positivity of the coefficients $F_s(\alpha, \beta)$ is already less than trivial when s is a fairly simple rational of $\mathbb{P}^1\mathbb{Q}$, say an integer (that case was used in Section 7 of [Gu], to establish a certain convergence property in the Teichmüller space of the cusped torus). In general, this author wonders about a possible interpretation (geometric, algebraic or combinatorial) of these positive numbers $F_s(\alpha, \beta)$.

2. The functions f_s are Laurent polynomials

Let $S = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ be the one-cusped (or once-punctured) torus and $\pi : \mathbb{R}^2 - \mathbb{Z}^2 \to S$ the natural projection. Denote by $\mathcal{C}$ the set of isotopy classes of simple closed curves in S that are not the loop around the cusp. If p, q are coprime integers and ℓ is a line in $\mathbb{R}^2$ of slope $s = q/p$ missing $\mathbb{Z}^2$, then $\pi(\ell)$ defines an element c of $\mathcal{C}$. We call $s \in \mathbb{P}^1\mathbb{Q}$ the *slope* of c, and write $\sigma(c) = s$. It is well-known that σ establishes a bijection $\mathcal{C} \xrightarrow{\sim} \mathbb{P}^1\mathbb{Q}$. The curve of slope s is denoted by c_s.

Consider the hyperbolic plane $\mathbb{H}^2$ with its natural boundary $\partial \mathbb{H}^2 = \mathbb{P}^1\mathbb{R}$. Whenever two curves $c, c' \in \mathcal{C}$ have (minimal) intersecion number 1, we connect the rationals $\sigma(c)$ and $\sigma(c')$ by a line in $\mathbb{H}^2$. The result is the *Farey triangulation* of $\mathbb{H}^2$ into infinitely many ideal *Farey triangles*. It is well-known that the triples of vertices of Farey triangles are exactly those triples of rationals that can be written

$$\left(\frac{q_0}{p_0} \ , \ \frac{q_0 + q_1}{p_0 + p_1} \ , \ \frac{q_1}{p_1} \right) \quad \text{where} \quad \left| \begin{pmatrix} q_0 & q_1 \\ p_0 & p_1 \end{pmatrix} \right| = \pm 1$$

(we agree that $\infty = \frac{\pm 1}{0}$). Geometrically, the Farey triangulation is generated by reflecting the triangle $1\infty 0$ in its sides *ad infinitum*.

Choose a point $p \in S$. Let τ be the trace operator on $SL_2(\mathbb{R})$, and fix a representation $\rho : \pi_1(S, p) \to SL_2(\mathbb{R})$ such that if $\gamma \in \pi_1(S, p)$ is in the conjugacy class of the loop around the puncture, then $\tau \circ \rho(\gamma) = -2$ (we say that ρ is *type-preserving*).

Proposition 2. *The trace τ induces a function, also noted τ, on $\mathcal{C} \simeq \mathbb{P}^1\mathbb{Q}$. If s, s_0, s_1, s' are elements of $\mathbb{P}^1\mathbb{Q}$ such that $s_0 s_1 s$ and $s_0 s_1 s'$ are Farey triangles, then $\tau(s)$ and $\tau(s')$ are the roots of the polynomial $X^2 - \tau(s_0)\tau(s_1)X + \tau(s_0)^2 + \tau(s_1)^2$.*

Proof. Defining τ on $\mathcal{C}$ is straightforward, since each curve in $\mathcal{C}$ determines a conjugacy class (together with its inverse) in the image of ρ. We will further omit the slope bijection $\sigma : \mathcal{C} \to \mathbb{P}^1\mathbb{Q}$ and simply consider τ as defined on $\mathbb{P}^1\mathbb{Q}$.

The modular group $SL_2(\mathbb{Z})$ acts naturally on the cusped torus S while preserving the isotopy class of the loop around the cusp. The induced action on $\mathcal{C}$ coincides (*via* σ) with the Möbius action on $\mathbb{P}^1\mathbb{Q} \subset \partial \mathbb{H}^2$, which extends to an action on the Farey triangulation of $\mathbb{H}^2$ that is transitive on the set of all Farey edges $s_0 s_1$.

Endow the two curves $c_{s_0}, c_{s_1} \in \mathcal{C}$ with orientations and arrange c_{s_0} and c_{s_1} in S so that they intersect only at the base point $p \in S$. Then c_{s_0}, c_{s_1} define elements g_{s_0}, g_{s_1} of $\pi_1(S, p)$.

Observation: $[g_{s_0}, g_{s_1}]$ determines a simple loop around the puncture, and therefore has trace -2. The curves c_s and $c_{s'}$ determine the conjugacy classes of $g_{s_0} g_{s_1}$ and $g_{s_0} g_{s_1}^{-1}$ (not necessarily in that order, depending on the chosen orientations).

This observation can be checked easily when $(s_0, s_1) = (0, \infty)$ (hence $\{s, s'\} = \{1, -1\}$). The general case follows because the curves in $\mathcal{C}$ which have intersection number 1 with c_{s_0} and c_{s_1} are always exactly c_s and $c_{s'}$, and the $SL_2(\mathbb{Z})$-action (transitive on Farey edges $s_0 s_1$) respects the intersection numbers and the loop around the cusp.

Recall the following trace relations, valid for all $a, b \in SL_2(\mathbb{R})$:

$$\tau(ab) + \tau(ab^{-1}) = \tau(a)\tau(b)$$
$$\tau(ab)\tau(ab^{-1}) = \tau^2(a) + \tau^2(b) - 2 - \tau([a, b]).$$

Setting $a = g_{s_0}$, $b = g_{s_1}$, the Proposition follows. $\qquad\square$

In the notation above, we now define $f_s := \tau(s)$. Dual to the Farey triangulation is an infinite 3-valent tree in $\mathbb{H}^2$ whose complementary regions R_s stand in bijection with the Farey vertices $s \in \mathbb{P}^1\mathbb{Q}$. The Markoff map Φ is therefore defined by $\Phi(R_s) = f_s$. By Proposition 2, the variables

$$(X, Y, Z) = (f_0, f_\infty, f_{-1})$$

of (1) satisfy the *Markoff equation*

$$X^2 + Y^2 + Z^2 = XYZ.$$

(This equation defines the character variety, or variety of type-preserving representations.) Moreover, Proposition 2 implies that if $(A, B, C, D) = (f_{s'}, f_{s_0}, f_{s_1}, f_s)$ and A, B, C are known (for example in terms of X, Y, Z), then we can always recover D by either one of the formulas

$$D = BC - A \qquad \text{or} \qquad D = (B^2 + C^2)/A.$$

In fact, these relations allow us to define f_s (and therefore Φ) inductively for all $s \in \mathbb{P}^1\mathbb{Q}$, in terms of X, Y, Z. In order to make each $f_s = \Phi(R_s)$ a homogeneous *Laurent polynomial* of degree 1 in X, Y, Z, we tweak the first induction relation above and use

$$(3) \qquad\qquad f_s = f_{s_0} f_{s_1} \frac{X^2 + Y^2 + Z^2}{XYZ} - f_{s'}$$

where s, s_0, s_1, s' are as in Proposition 2. For example, $f_1 = \frac{X^2 + Y^2}{Z}$. For all $s \in \mathbb{P}^1\mathbb{Q}$, denote by $[s]$ the unique element of $\{0, -1, \infty\}$ such that s and $[s]$ project to the same point of $\mathbb{P}^1(\mathbb{Z}/2\mathbb{Z})$. In particular, $f_{[s]}$ is one of the variables X, Y, Z.

Proposition 3. *If f_s is defined inductively for all $s \in \mathbb{P}^1\mathbb{Q}$ using (3), then f_s is a Laurent polynomial in X, Y, Z. Moreover there is a finitely supported function $F_s : \mathbb{Z}^2 \to \mathbb{Z}$ such that*

$$f_s = \left(\sum_{\alpha, \beta \in \mathbb{Z}} F_s(\alpha, \beta) \frac{X^{1+\alpha} Y^{1+\beta}}{Z^{1+\alpha+\beta}} \right) \in f_{[s]} \cdot \mathbb{Z}\left[X^{\pm 2}, Y^{\pm 2}, Z^{\pm 2} \right].$$

Proof. From (3), by induction, f_s is a Laurent polynomial. The claim on the parity of the degrees also follows by induction from (3), because $\{f_{[s_0]}, f_{[s_1]}, f_{[s]}\} = \{X, Y, Z\} = \{f_{[s_0]}, f_{[s_1]}, f_{[s']}\}$ holds whenever s, s_0, s_1, s' are as in Proposition 2.   □

In Section 3 we prove Theorem 1 for positive rationals s. The remaining cases ($s < -1$ and $-1 < s < 0$) will follow by a symmetry argument (see Section 4). Section 5 exposes a generalization of our "tweaking" operation (3), and a conjecture extending Theorem 1.

3. A FAMILY OF DOMAINS AND FUNCTIONS

Define $\mathcal{Q} = \mathbb{Q}^{\geq 0} \cup \{\infty\}$. Any point s of $\mathcal{Q}$ can be written in a unique way

$$s = \frac{q}{p} \quad \text{with } p, q \in \mathbb{N} \text{ coprime}$$

(we agree that $\infty = \frac{1}{0}$). For such $s \in \mathcal{Q}$, define

$$(4) \qquad J_s := \left\{ (\alpha, \beta) \in \mathbb{Z}^2 \;\middle|\; \begin{array}{l} \alpha \equiv q \;;\; \beta \equiv p \; [2] \\ \alpha \geq -q \;;\; \beta \geq -p \\ \alpha + \beta \leq p + q - 2 \\ p\alpha + q\beta \geq 0 \end{array} \right\}.$$

It will turn out that F_s is supported exactly on J_s. Observe that $J_0 = \{(0, -1)\}$ and $J_\infty = \{(-1, 0)\}$ and $J_1 = \{(-1, 1); (1, -1)\}$. Further, define

- $Z_s = (q, p) + 2\mathbb{Z}^2$ so that $J_s \subset Z_s$;
- $P_i^s = (q + 2i, -p) \in Z_s$ for all $i \in \mathbb{Z}$;
- $Q_j^s = (-q, p + 2j) \in Z_s$ for all $j \in \mathbb{Z}$;
- $\varphi_s(\alpha, \beta) = p\alpha + q\beta$;
- $\Lambda = \{(0, 0); (0, 2); (2, 0)\}$;
- $n\Lambda = \Lambda + \cdots + \Lambda = \{(2i, 2j) \in 2\mathbb{N}^2 \,|\, i + j \leq n\}$ for all $n \in \mathbb{N}$;
- If U is a subset of Z_s, then $\langle U \rangle_s$ denotes the intersection with Z_s of the convex hull of U in $\mathbb{R}^2$.
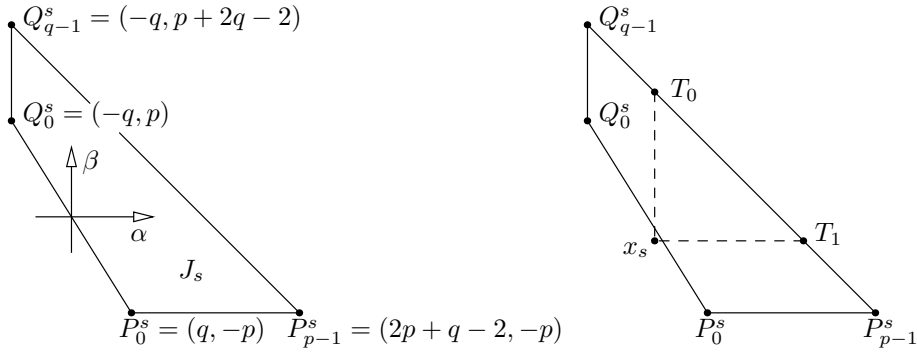

FIGURE 1. The domain J_s.

Lemma 4. *For all s in $\mathcal{Q}$, one has $P_{p-1}^s, Q_{q-1}^s \in J_s$ and*

$$J_s = \left\langle \{P_i^s \,|\, 0 \leq i < p\} \cup \{Q_j^s \,|\, 0 \leq j < q\} \right\rangle_s.$$

Proof. Having checked the two cases $s = 0, \infty$ separately (one of the families $\{P_i^s\}, \{Q_j^s\}$ is then empty, so the second statement does not imply the first), assume $p, q \geq 1$ and focus on the second statement. Observe that $P_{p-1}^s, P_0^s, Q_0^s, Q_{q-1}^s$ are (in that order) the extremal points of a convex quadrilateral (or triangle, or segment, when $p = 1$ and/or $q = 1$), as shown in Figure 1 (left). The sides of the quadrilateral correspond to the four inequalities defining J_s, hence the result. $\square$

Corollary 5. *For all s in $\mathcal{Q}$ and n in $\mathbb{N}$, one has*

$$J_s + n\Lambda = \left\langle \{P_i^s | 0 \leq i < p + n\} \cup \{Q_j^s | 0 \leq j < q + n\}\right\rangle_s$$
$$J_s + \Lambda \supset [P_0^s + p\Lambda] \cup [Q_0^s + q\Lambda].$$

Proof. Again, check the cases $s = 0, \infty$ separately. If $p, q \geq 1$, the first statement follows easily from Lemma 4 (which covers the case $n = 0$), and the second follows from the first (with $n = 1$) by observing that $P_0^s + p\Lambda$ and $Q_0^s + q\Lambda$ are the convex hulls of points of $J_s + \Lambda$: for instance,

$$P_0^s + p\Lambda = \left\langle \{P_0^s; P_p^s; (q, p)\}\right\rangle_s$$
$$= \left\langle \left\{P_0^s; P_p^s; \frac{qP_p^s + pQ_q^s}{q + p}\right\}\right\rangle_s.$$

$\square$

We now redefine the coefficient functions $F_s(\cdot, \cdot)$ of Proposition 3 from a slightly altered point of view. Let $\mathcal{F}$ be the $\mathbb{Z}$-module of functions $F : \mathbb{Z}^2 \to \mathbb{Z}$ having finite support. We can define a convolution law on $\mathcal{F}$ by $F * G(u) = \sum_{x+y=u} F(x)G(y)$. Also, denoting by $\mathbb{1}_U$ the characteristic function of a set U, define the following elements of $\mathcal{F}$:

$$F_s = \mathbb{1}_{J_s} \text{ for } s \in \{0, 1, \infty\} .$$

It is straightforward to check that the identity of Proposition 3 holds for $s \in \{0, 1, \infty\}$. Finally, for $s \in \mathcal{Q} - \{0, 1, \infty\}$, we shall define F_s in an inductive way. In $\mathbb{H}^2$ endowed with the Farey triangulation, consider the line L_s connecting s to the midpoint $\sqrt{-1}$ of the line 0∞. Denote by s_0, s_1 the ends of the first Farey edge encountered by L_s (closest to s). We call s_0 and s_1 the *parents* of s. Up to exchanging indices, we may assume that the parents of s_1 are s_0 and another point $s' \in \mathcal{Q}$ (we agree that the parents of 1 are 0 and ∞). See Figure 2. In particular, one has

$$(5) \qquad \left.\begin{array}{rcl} (p, q) & = & (p_1, q_1) + (p_0, q_0) \\ (p', q') & = & (p_1, q_1) - (p_0, q_0) \end{array}\right\} \text{ for } (s, s', s_0, s_1) = (\tfrac{q}{p}, \tfrac{q'}{p'}, \tfrac{q_0}{p_0}, \tfrac{q_1}{p_1}) .$$

Definition 6. For each configuration as above, we set

$$(6) \qquad F_s := (F_{s_0} * F_{s_1} * \mathbb{1}_\Lambda) - F_{s'} \quad \text{where } \Lambda = \{(0,0); (0,2); (2,0)\}.$$

Since the dual of the Farey triangulation is a tree, this definition is easily seen to be consistent. Clearly, F_s is in $\mathcal{F}$. It is easy to check that (6) is just a reformulation of (3), so (6) agrees with our first definition (Prop. 3) of F_s. The following three Lemmas (numbered 7-8-9) are intended to prove that F_s is supported on J_s and $F_s(J_s) > 0$, for all $s \in \mathcal{Q}$. The reader is invited to read their three statements first (the three proofs could be written as one vast simultaneous induction on s for the simultaneous three statements).

Lemma 7. *For each configuration as above where $s \in \mathcal{Q} - \{0, 1, \infty\}$, the set $J_{s'} \smallsetminus J_s$ consists of a unique (extremal) point x_s of $J_{s'}$, and $J_{s_0} + J_{s_1} + \Lambda = J_s \sqcup \{x_s\}$.*

Remark: if $s \in \mathcal{Q} - \{0, \infty\}$, following Lemma 4, we call "extremal" the points $P_0^s, P_{p-1}^s, Q_0^s, Q_{q-1}^s$ of J_s (with possible repeats). If $s \in \{0, \infty\}$, then J_s is reduced to an (extremal) point $P_{p-1}^s = Q_{q-1}^s$.

Proof. Let (α, β) be an element of $J_{s'}$. By (5) one has $Z_{s'} = Z_s$ so (α, β) satisfies the congruence conditions of (4). Still by (5), one has $p' \leq p$ and $q' \leq q$ so the first three inequalities of (4) are also satisfied at (α, β). For the fourth inequality, consider the linear form $\varphi_s(\alpha, \beta) = p\alpha + q\beta$. Clearly, $\varphi_s(Z_s) \subset 2\mathbb{Z}$. Furthermore, observe

$$
\begin{aligned}
\varphi_s(P_i^{s'}) &= pq' - qp' + 2ip \\
\varphi_s(Q_j^{s'}) &= qp' - pq' + 2jq \\
pq' - qp' &= 2(p_0 q_1 - p_1 q_0) = \pm 2 \ (s_0, s_1 \text{ Farey neighbors}).
\end{aligned}
$$

Thus, if $p' = 0$ (resp. $q' = 0$), taking for x_s the only point $Q_0^{s'}$ (resp. $P_0^{s'}$) of $J_{s'}$ yields $\varphi_s(x_s) = -2$. If $p'q' > 0$, we find that exactly one point x_s among $\{P_0^{s'}, Q_0^{s'}\}$ satisfies $\varphi_s(x_s) = -2$ while $\varphi_s(x) \geq 0$ at all other extremal points x of $J_{s'}$. It follows that on $J_{s'} - \{x_s\}$ one has $\varphi_s > -2$ i.e. $\varphi_s \geq 0$. Hence the first statement.

Let us now prove the second statement. For $(y_0, y_1, \lambda) \in J_{s_0} \times J_{s_1} \times \Lambda$, it is again straightforward to check that $(\alpha, \beta) = y_0 + y_1 + \lambda$ satisfies the congruence conditions and the first three inequalities of (4). For the fourth, compute

$$
\begin{array}{ll}
\varphi_s(P_i^{s_0}) = p_1 q_0 - p_0 q_1 + 2ip & \varphi_s(P_i^{s_1}) = p_0 q_1 - p_1 q_0 + 2ip \\
\varphi_s(Q_j^{s_0}) = p_0 q_1 - p_1 q_0 + 2jq & \varphi_s(Q_j^{s_1}) = p_1 q_0 - p_0 q_1 + 2jq.
\end{array}
$$

Again, observe that $p_0 q_1 - p_1 q_0 = \pm 1$. The same argument as above (involving this time extremal points of J_{s_0}, J_{s_1} instead of $J_{s'}$) shows that φ_s takes the value -1 at exactly one point $y_0 \in \{P_0^{s_0}, Q_0^{s_0}\}$ (resp. $y_1 \in \{P_0^{s_1}, Q_0^{s_1}\}$) and $\varphi_s \geq 1$ holds on $J_{s_0} - \{y_0\}$ (resp. $J_{s_1} - \{y_1\}$). Moreover, y_k belongs to J_{s_k} for $k \in \{0, 1\}$ (this is immediate from Lemma 4, unless $p_k q_k = 0$ where we need to check separately). The following table summarizes the two possible cases for y_0, y_1, x_s.

(7)

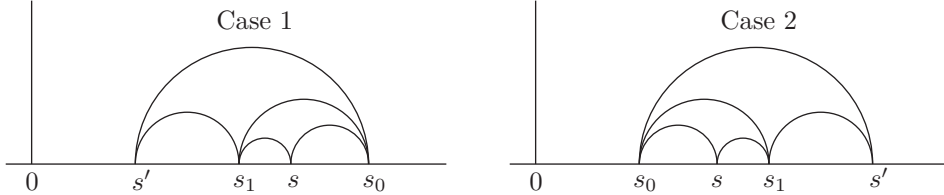|	$p_0 q_1 - p_1 q_0$	y_0	y_1	x_s
Case 1	-1	$Q_0^{s_0}$	$P_0^{s_1}$	$P_0^{s'}$
Case 2	1	$P_0^{s_0}$	$Q_0^{s_1}$	$Q_0^{s'}$

FIGURE 2.

Using Relations (5) and the definitions of P_i^s and Q_j^s, one checks immediately that $y_0 + y_1 = x_s$ in both cases. Since φ_s is linear, x_s turns out to be the only

point of $J_{s_0} + J_{s_1} + \Lambda$ where $\varphi_s < 0$. This gives one inclusion of the equality to be proved.

For the other inclusion, $J_s \sqcup \{x_s\} \subset J_{s_0} + J_{s_1} + \Lambda$, we shall restrict to Case 1 above (Case 2 is similar). By Table (7), since $Q_0^{s_0}$ and $P_0^{s_1}$ belong to J_{s_0} and J_{s_1}, one has $q_0, p_1 > 0$. In view of Corollary 5, it is sufficient to prove that

$$(8) \qquad J_s \sqcup \{x_s\} \subset (J_{s_0} + P_0^{s_1} + p_1\Lambda) \cup (J_{s_1} + Q_0^{s_0} + q_0\Lambda).$$

Still by Corollary 5, since $P_0^{s_1} + Z_{s_0} = Z_s$, one has

$$\begin{aligned}
J_{s_0} + P_0^{s_1} + p_1\Lambda &= \left\langle P_0^{s_1} + \left(\{P_i^{s_0}|0 \leq i < p_0 + p_1\} \cup \{Q_j^{s_0}|0 \leq j < q_0 + p_1\}\right)\right\rangle_s \\
&= \left\langle \{P_i^s|0 \leq i < p\} \cup \{Q_0^{s_0} + P_0^{s_1}, Q_{q_0+p_1-1}^{s_0} + P_0^{s_1}\}\right\rangle_s \\
&= \left\langle \{P_i^s|0 \leq i < p\} \cup \{x_s, T_0\}\right\rangle_s \\
\text{where } T_0 &= (q - 2q_0, p + 2(q_0 - 1)).
\end{aligned}$$

(To write the second line, we replaced the collection of the $Q_j^{s_0}$ by its extremal terms: this is justified because $q_0 + p_1 > 0$). Similarly,

$$\begin{aligned}
J_{s_1} + Q_0^{s_0} + q_0\Lambda &= \left\langle \{Q_j^s|0 \leq j < q\} \cup \{x_s, T_1\}\right\rangle_s \\
\text{where } T_1 &= (q + 2(p_1 - 1), p - 2p_1).
\end{aligned}$$

We just captured all the P_i^s, Q_j^s which according to Lemma 4 define J_s (Figure 1, right). Observe that T_0 (resp. T_1) has the same abscissa (resp. ordinate) as $x_s = (q', -p')$. Finally, the facts that the points $Q_{q-1}^s, T_0, T_1, P_{p-1}^s$ lie in that order on the edge $E = Q_{q-1}^s P_{p-1}^s$ of J_s, and that the edge $P_0^s Q_0^s$ of J_s (defined by "$\varphi_s = 0$") separates x_s from E, imply (8). See the right panel of Figure 1. $\qquad\square$

Lemma 8. *The function F_s is supported on a subset of J_s for all $s \in \mathcal{Q}$, and if c is an extremal point of J_s, then $F_s(c) = 1$.*

Proof. We prove both facts by simultaneous induction. They hold for $s \in \{0, 1, \infty\}$ so assume they hold for s_0, s_1, s' and let us prove them for s. By (6), F_s is supported on $(J_{s_0} + J_{s_1} + \Lambda) \cup J_{s'} = J_s \cup \{x_s\}$, with x_s defined as in Lemma 7. Recall the linear form φ_s from the proof of Lemma 7: over $J_{s_0}, J_{s_1}, \Lambda$, the form φ_s achieves its respective minima only at the extremal points $y_0, y_1, 0$; therefore x_s is realized in $J_{s_0} + J_{s_1} + \Lambda$ only as $y_0 + y_1 + (0, 0)$. Hence, by induction, $F_{s_0} * F_{s_1} * \mathbb{1}_\Lambda(x_s) = 1$. But x_s is also an extremal point of $J_{s'}$, so (6) yields $F_s(x_s) = 0$: the function F_s is supported within J_s.

Next, observe that the extremal point P_{p-1}^s of J_s maximizes the first coordinate (a similar statement is true for $J_{s_0}, J_{s_1}, J_{s'}$). Since $P_{p-1}^s = P_{p_0-1}^{s_0} + P_{p_1-1}^{s_1} + (2, 0)$, one has by induction $F_{s_0} * F_{s_1} * \mathbb{1}_\Lambda(P_{p-1}^s) = 1$. Also, P_{p-1}^s does not belong to $J_{s'}$ because all (α, β) in $J_{s'}$ satisfy $\alpha + \beta \leq p' + q' - 2 < p + q - 2$. By (6), we find $F_s(P_{p-1}^s) = 1$. Similarly, $F_s(Q_{q-1}^s) = 1$. Consider one of the (at most two) remaining extremal points of J_s, say P_0^s. Without loss of generality, one has $p \geq 2$ (otherwise, the point has already been treated as P_{p-1}^s). One cannot have $\{p_0, p_1\} = \{0, p\}$ lest $|p_0 q_1 - p_1 q_0| \geq p > 1$ (recall s_0, s_1 are Farey neighbors). Therefore $p_0, p_1 \geq 1$. Observe that the points $P_0^s, P_0^{s_0}, P_0^{s_1}$ are the minimizers over J_s, J_{s_0}, J_{s_1} of the form $(\alpha, \beta) \mapsto \beta + \varepsilon\alpha$, for very small ε. Since $P_0^s = P_0^{s_0} + P_0^{s_1} + (0, 0)$, we find that $F_{s_0} * F_{s_1} * \mathbb{1}_\Lambda(P_0^s) = 1$. Finally, P_0^s cannot belong to $J_{s'}$ because of its second coordinate, $-p < -p'$. By (6), this yields $F_s(P_0^s) = 1$. Similarly, $F_s(Q_0^s) = 1$. $\qquad\square$

Lemma 9. *For all $s \in \mathcal{Q}$ one has $F_s(J_s) \subset \mathbb{Z}^{>0}$. If $s \notin \{0, \infty\}$ then*

$$\mathbb{1}_{J_s} \cdot \sup \left\{ \begin{array}{ccc} \mathbb{1}_{\{P_0^{s_0}\}} * F_{s_1} & , & \mathbb{1}_{\{P_0^{s_1}\}} * F_{s_0} \ , \\ \mathbb{1}_{\{Q_0^{s_0}\}} * F_{s_1} & , & \mathbb{1}_{\{Q_0^{s_1}\}} * F_{s_0} \end{array} \right\} \leq F_s.$$

Remark 10. By Corollary 5 and Lemma 7, each function in the bracket is supported within $J_s \sqcup \{x_s\}$ (because e.g. $P_0^{s_0} \in J_{s_0} + \Lambda$). In other words, $\mathbb{1}_{J_s}$ can be replaced by $\mathbb{1}_{Z_s - \{x_s\}}$ without altering the strength of the statement.

Proof. Again, both facts are proved by simultaneous induction. They hold for $s \in \{0, 1, \infty\}$; assume they hold for s_0, s_1, s'; let us prove them for s. Recall our convention that the parents of s_1 are s_0 and s' (so in particular, $s_1 \neq 0, \infty$). We saw in the course of proving Lemma 7 that x_s is either $P_0^{s_0} + Q_0^{s_1} = Q_0^{s'}$ or $Q_0^{s_0} + P_0^{s_1} = P_0^{s'}$. On the other hand, x_{s_1} is either $P_0^{s_0} + Q_0^{s'}$ or $Q_0^{s_0} + P_0^{s'}$. In fact, using (5) and the generic characterization $\varphi_\sigma(x_\sigma) = -2$, it is easy to check that

$$(9) \qquad \begin{array}{ccccc} x_{s_1} = P_0^{s_0} + Q_0^{s'} & \iff & q_0 p' - p_0 q' = -1 & \iff & x_s = Q_0^{s'} \ ; \\ x_{s_1} = Q_0^{s_0} + P_0^{s'} & \iff & p_0 q' - q_0 p' = -1 & \iff & x_s = P_0^{s'} \ . \end{array}$$

Define in general $G_s = F_s * \mathbb{1}_\Lambda$. Lemma 8 easily yields $G_\sigma(P_0^\sigma) = G_\sigma(Q_0^\sigma) = 1$ for all $\sigma \in \mathcal{Q}$ (this should again be checked separately for $\sigma = 0, \infty$). By Lemma 7 and the induction hypothesis, we have $F_{s_0} * F_{s_1} * \mathbb{1}_\Lambda > 0$ on J_s. Moreover, by (6),

$$\begin{aligned} F_s + F_{s'} &=& F_{s_0} * F_{s_1} * \mathbb{1}_\Lambda \\ &=& \sum_{\lambda \in (J_{s_0} + \Lambda)} G_{s_0}(\lambda) \cdot \mathbb{1}_{\{\lambda\}} * F_{s_1} \\ &=& \left[\left(\mathbb{1}_{\{P_0^{s_0}\}} + \mathbb{1}_{\{Q_0^{s_0}\}} \right) * F_{s_1} \right] + \sum_{\substack{\lambda \in (J_{s_0} + \Lambda) \\ \lambda \neq P_0^{s_0}, Q_0^{s_0}}} G_{s_0}(\lambda) \cdot \mathbb{1}_{\{\lambda\}} * F_{s_1} \ . \end{aligned}$$

Thus, if we prove

$$(10) \qquad \mathbb{1}_{\{P_0^{s_0}\}} * F_{s_1}(x) \geq F_{s'}(x) \ ; \ \mathbb{1}_{\{Q_0^{s_0}\}} * F_{s_1}(x) \geq F_{s'}(x) \ \text{ for all } x \neq x_s \ ,$$

then we will have at once $F_s > 0$ on J_s (because $F_s + F_{s'} \geq 2 F_{s'}$ and $F_{s'}(J_{s'}) > 0$), and also $F_s \geq \sup \left\{ \mathbb{1}_{\{P_0^{s_0}\}} * F_{s_1}, \mathbb{1}_{\{Q_0^{s_0}\}} * F_{s_1} \right\}$ on J_s. That is half of Lemma 9.

Using the relation $P_0^{s_0} = -Q_0^{s_0}$ and the identities $\mathbb{1}_{\{\xi\}} * \mathbb{1}_{\{\eta\}} = \mathbb{1}_{\{\xi + \eta\}}$ and $\mathbb{1}_{\{\xi\}} * f(x + \xi) = f(x)$, Equation (10) is equivalent to

$$(11) \qquad\qquad F_{s_1}(y) \geq \mathbb{1}_{\{Q_0^{s_0}\}} * F_{s'}(y) \ \text{ if } \ y \neq x_s + Q_0^{s_0}$$

$$(12) \qquad\qquad F_{s_1}(y) \geq \mathbb{1}_{\{P_0^{s_0}\}} * F_{s'}(y) \ \text{ if } \ y \neq x_s + P_0^{s_0}.$$

For $y \neq x_{s_1}$, both inequalities are already true by induction (s_0, s' are the parents of s_1). For $y = x_{s_1}$, in view of (9), two cases may arise:

- If $x_s = P_0^{s'}$ then $x_{s_1} = x_s + Q_0^{s_0}$ so (11) is true, and (12) need only be checked at $y = x_{s_1}$. One has $F_{s_1}(x_{s_1}) = 0$ and

$$\mathbb{1}_{\{P_0^{s_0}\}} * F_{s'}(x_{s_1}) = F_{s'}(x_{s_1} - P_0^{s_0}) = F_{s'}(P_0^{s'} + 2Q_0^{s_0}).$$

  However, (5) yields $\varphi_{s'}(P_0^{s'} + 2Q_0^{s_0}) = 2(p_0 q' - q_0 p') = -2$; hence, the point $(P_0^{s'} + 2Q_0^{s_0})$ does not belong to $J_{s'}$ and $\mathbb{1}_{\{P_0^{s_0}\}} * F_{s'}(x_{s_1}) = 0$.

- Similarly, if $x_s = Q_0^{s'}$ then (12) is true, and for (11) one need only check $\mathbb{1}_{\{Q_0^{s_0}\}} * F_{s'}(x_{s_1}) = F_{s'}(Q_0^{s'} + 2P_0^{s_0}) = 0$ because $\varphi_{s'}(Q_0^{s'} + 2P_0^{s_0}) = -2 < 0$.

It remains to prove $F_s \geq \sup\left\{\mathbb{1}_{\{P_0^{s_1}\}} * F_{s_0}, \mathbb{1}_{\{Q_0^{s_1}\}} * F_{s_0}\right\}$ on J_s. By the lower bounds on F_s we just established, it is enough to make sure

(13) $\qquad \mathbb{1}_{\{P_0^{s_1}\}} * F_{s_0} \leq \mathbb{1}_{\{P_0^{s_0}\}} * F_{s_1} \ ; \ \mathbb{1}_{\{Q_0^{s_1}\}} * F_{s_0} \leq \mathbb{1}_{\{Q_0^{s_0}\}} * F_{s_1} \quad \text{on } Z_s - \{x_s\}.$

We focus only on the first inequality (the second is similar). It is equivalent, by the same method as above, to:

$$\mathbb{1}_{\{P_0^{s'}\}} * F_{s_0}(y) \leq F_{s_1}(y) \ \text{ if } y \neq x_s + Q_0^{s_0}$$

(we used $P_0^{s'} = P_0^{s_1} + Q_0^{s_0}$, a consequence of (5)). But that inequality is true (by induction) as long as $y \neq x_{s_1}$. Again, in view of (9), two cases may arise at $y = x_{s_1}$:

- If $x_s = P_0^{s'}$ then $x_{s_1} = x_s + Q_0^{s_0}$ and there is nothing to do;
- If $x_s = Q_0^{s'}$ we only need check the inequality above at $y = x_{s_1}$. On one hand, $F_{s_1}(x_{s_1}) = 0$; on the other,

$$\mathbb{1}_{\{P_0^{s'}\}} * F_{s_0}(x_{s_1}) = F_{s_0}(x_{s_1} - P_0^{s'}) = F_{s_0}(P_0^{s_0} + 2Q_0^{s'})$$

but, by (5), $\varphi_{s_0}(P_0^{s_0} + 2Q_0^{s'}) = 2(q_0 p' - p_0 q') = -2 < 0$ so the point $(P_0^{s_0} + 2Q_0^{s'})$ does not belong to J_{s_0} and $\mathbb{1}_{\{P_0^{s'}\}} * F_{s_0}(x_{s_1}) = 0$.

Theorem 1 is proved for all $s \in \mathcal{Q}$. $\hfill\square$

4. Formal Markoff map

Figure 3 shows the domains J_s and the values of F_s for some of the simplest rationals $s \in \mathcal{Q}$. In each case, the points x of the affine lattice Z_s have been identified with the cells of a honeycomb, carrying the numbers $F_s(x)$. Empty cells carry 0, by convention. Coordinates have been tilted so that the edge $P_0^s Q_0^s$ of J_s is always at the top of J_s, rather than the bottom left as in Figure 1. The left edge of J_s consists of p cells (the P_i^s); the right edge, of q cells (the Q_j^s). The single cells to the bottom left and bottom right of the "root" (dark spot) correspond to the exceptional cases $s = 0$ and $s = \infty$. The single cell above the root corresponds to $s = -1$; the meaning of that convention, already apparent from the Introduction, will be re-emphasized in a moment. Observe the $1's$ in the corners of each J_s, just as in Lemma 8. It is an easy exercise (left to the reader) to prove by induction that the bottom, left, and right edges of each J_s (for $s \in \mathcal{Q} - \{0, \infty\}$) always carry full lines of the Pascal triangle: if $v = (2, -2)$ then

$$F_s(P_i^s) = \binom{p-1}{i}; \ F_s(Q_j^s) = \binom{q-1}{j}; \ F_s(Q_{q-1}^s + kv) = \binom{p+q-1}{k}.$$

Notice the arrangement of the various J_s in the complement U of a planar 3-valent tree: this tree should be seen as the dual of the Farey triangulation of $\mathbb{H}^2$, so each connected component R_s of U corresponds to a horosphere centered at a rational point s. Each configuration s, s_0, s_1, s' as in the previous section corresponds in fact to a pair of edge-adjacent components R_{s_0}, R_{s_1} of U, together with their two common neighbors $R_s, R_{s'}$. Since Formula (6) is symmetric in s, s', one may apply it backwards to define F_s for all s in $\mathbb{P}^1\mathbb{Q}$ (not just $\mathcal{Q}$). This was (very) partially done in Figure 3 by showing $J_{-1} = \{(-1, -1)\}$ just above the root. However, the full picture would exhibit a 6-fold dihedral symmetry around the root, so only one sixth of the tree is explored to some depth in Figure 3. This 6-fold symmetry is also the reason why honeycombs were used instead of, say, square
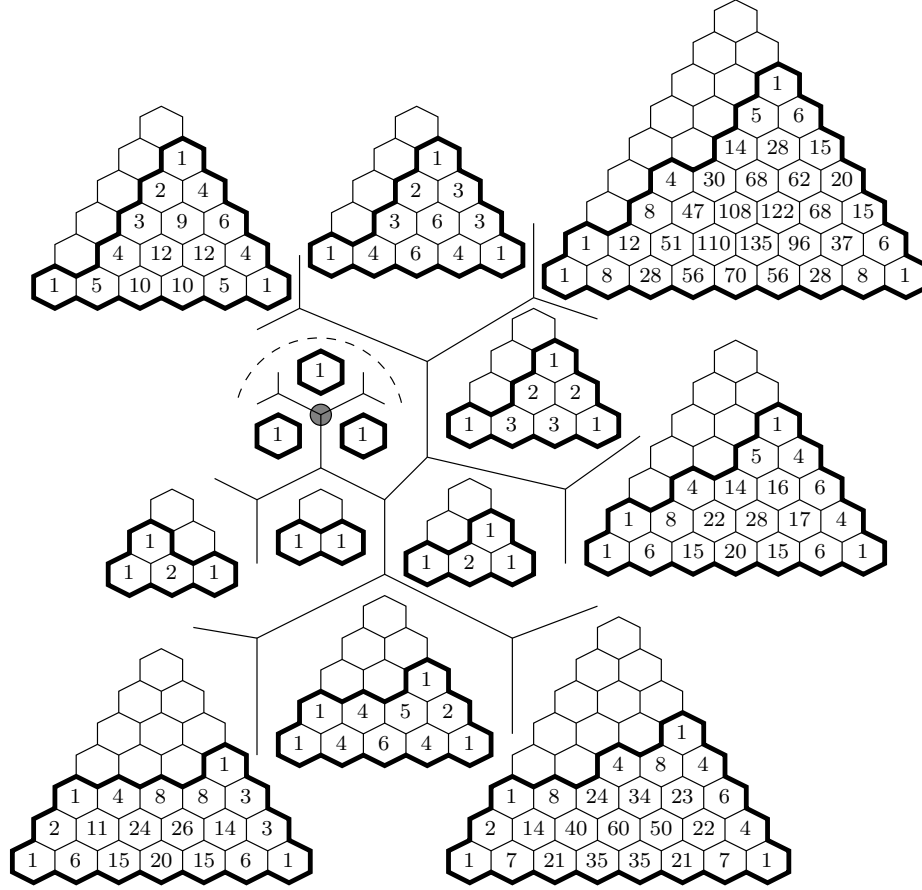
FIGURE 3. The universal (formal) Markoff map. The integers $F_s(\cdot, \cdot)$ inside each "bag" add up to a Markoff number.

cells. As an exercise, the reader may prove the following formulas for the symmetry (true for all $s \in \mathbb{P}^1\mathbb{Q}$) by induction on the tree:

$$F_{\frac{1}{s}}(\alpha, \beta) = F_s(\beta, \alpha) \; ; \; F_{-1-s}(\alpha, \beta) = F_s(-2 - \alpha - \beta, \beta)$$

(The Möbius transformations acting on the index s permute the rationals $-1, 0, \infty$ while the affine transformations acting on the argument (α, β) permute the associated singletons J_{-1}, J_0, J_∞, as well as the elements of $-\Lambda$).

5. CONJECTURAL GENERALIZATION

The Markoff polynomial $M = X^2 + Y^2 + Z^2 - XYZ$ encountered in Section 2 has degree 2 in each variable. This is why any solution (X, Y, Z) of the equation $M = 0$ defines many other solutions: by considering M as a polynomial of degree 2 in, say, the variable X, we can always replace X by the conjugate root. Thus, the free product G of three copies of $\mathbb{Z}/2\mathbb{Z}$ acts naturally on the variety $M = 0$ by isomorphisms. An analogous statement holds true if we replace M by *any*

polynomial of degree 2 in all its variable (allowing for such monomials as X^2Y^2ZT), and allow for actions by birational isomorphisms.

In this section, we conjecture a generalization of Theorem 1 to all N-variable polynomials M which are *monic of degree* 2 in each variable. Namely, we show that certain expressions for the action of G are Laurent polynomials (as in Proposition 3), and conjecture that the coefficients are positive. The coefficients of M will be considered as variables themselves (noted A_I below). We work over the complex field $\mathbb{C}$.

Let $N \geq 2$ be an integer, and denote by $[\![N]\!]$ the set of integers $\{1, 2, \ldots, N\}$. For each $I \subset [\![N]\!]$, fix a formal parameter A_I. Consider the Markoff-type equation in N variables $X_1 \ldots, X_N$:

$$\text{(14)} \qquad \sum_{i=1}^{N} X_i^2 + \sum_{I \subset [\![N]\!]} A_I \prod_{i \in I} X_i = 0 \ .$$

Let $V \subset \mathbb{C}^N$ be the variety defined by (14). For each $k \in [\![N]\!]$ and each point $(x_1, \ldots, x_N)$ of $V \cap \mathbb{C}^{*N}$, define

$$\text{(15)} \qquad \begin{aligned} E_k(x_1, \ldots, x_N) &:= (x_1, \ldots, x_{k-1}, \overline{x_k}, x_{k+1}, \ldots, x_n) \\[2mm] \text{where} \qquad \overline{x_k} &= \left(\sum_{i \neq k} x_i^2 + \sum_{I \subset [\![N]\!] - \{k\}} A_I \prod_{i \in I} x_i \right) \Big/ x_k \ . \end{aligned}$$

Then E_k defines a birational $\mathbb{Z}/2\mathbb{Z}$-action on V: indeed, $\overline{x_k} x_k$ is the product of the roots of (14), seen as a monic degree 2 polynomial in the k-th variable. By letting k range over $[\![N]\!]$, we obtain a birational action on V by the free product G of N copies of $\mathbb{Z}/2\mathbb{Z}$.

Observe that the variable $A_{[\![N]\!]}$ is absent from the definition (15) of each generator E_k: therefore, G acts on each "level manifold" of $\mathbb{C}^N$ defined by

$$\text{(16)} \qquad B(x_1, \ldots, x_N) := \left(\sum_{i=1}^{N} x_i^2 + \sum_{I \subsetneq [\![N]\!]} A_I \prod_{i \in I} x_i \right) \Big/ \prod_{i=1}^{N} x_i = \text{constant}$$

(indeed, $B(x_1, \ldots, x_N)$ is just the value of $A_{[\![N]\!]}$ for which a given point $(x_1, \ldots, x_N)$ will satisfy (14), when all the $\{A_I\}_{I \subsetneq [\![N]\!]}$ are given). In particular, $B(x_1, \ldots, x_N)$ is invariant under the action of E_k on $\mathbb{C}^N$: therefore, the expression given in (15) for E_k extends to a birational involution of $\mathbb{C}^N$ respecting B. Henceforward, we consider G as acting on $\mathbb{C}^N$ by birational isomorphisms.

Proposition 11. *For each g in G and $x = (x_1, \ldots, x_N)$ in $\mathbb{C}^N$, the coordinates of $g \cdot x$ are polynomials in the variables $\{x_i^{\pm 1}\}_{i \in [\![N]\!]}$ and $\{A_I\}_{I \subsetneq [\![N]\!]}$ with integer coefficients depending only on g.*

Remark 12. We conjecture that these integers are positive. Theorem 1 corresponds to $N = 3$ under the specialization $A_I \equiv 0$: for example, $E_1(x, y, z) = (\frac{y^2 + z^2}{x}, y, z)$.

Proof. We work by induction in G, using the generators E_k. When g is the identity of G, we are done. Suppose the proposition is true for g, so that $g \cdot (x_1, \ldots, x_N) = (y_1, \ldots, y_N)$ where each y_j is a polynomial in the $\{x_i^{\pm}\}_{i \in [\![N]\!]}$ and $\{A_I\}_{I \subsetneq [\![N]\!]}$ with integer coefficients. We must prove that the coordinates of

$$E_k(y_1, \ldots, y_N) = (y_1, \ldots, \overline{y_k}, \ldots, y_N)$$

are polynomials as well, where $\overline{y_k}$ is given as in (15). We saw that the left member $B(x_1, \ldots, x_N)$ of (16) is (formally) E_k-invariant for each $k \in [\![N]\!]$; therefore we must have $B(x_1, \ldots, x_N) = B(y_1, \ldots, y_N)$. Using (15), note that

$$
\begin{aligned}
\overline{y_k} &= \left(\sum_{i \neq k} y_i^2 + \sum_{I \subset [\![N]\!] - \{k\}} A_I \prod_{i \in I} y_i \right) \Big/ y_k \\
&= \left(\left(B(y_1, \ldots, y_N) \prod_{i=1}^{N} y_i \right) - y_k^2 - \sum_{\substack{I \subsetneq [\![N]\!] \\ k \in I}} A_I \prod_{i \in I} y_i \right) \Big/ y_k \\
&= B(x_1, \ldots, x_N) \left(\prod_{i \in [\![N]\!] - \{k\}} y_i \right) - y_k - \sum_{\substack{I \subsetneq [\![N]\!] \\ k \in I}} A_I \prod_{i \in I - \{k\}} y_i
\end{aligned}
$$

Using the formula (16) for $B(x_1, \ldots, x_N)$, the last expression is clearly a polynomial in the variables $\{x_i^{\pm 1}\}_{i \in [\![N]\!]}$ and $\{A_I\}_{I \subsetneq [\![N]\!]}$ with integer coefficients. This is a direct analogue of (3). $\qquad \square$

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